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Recursive Algorithm with Efficient Variables for Flexible Multibody Dynamics with Multiloop Constraints

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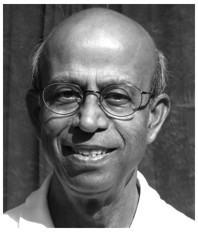
An algorithm is given to reduce computer time for simulation of motion of hinge-connected flexible multibody systems with multiple structural loops. The algorithm is based on using efficient motion variables for elastic motion and hinge rotation in a recursive formulation for an articulated system of bodies in a tree configuration. This formulation is then used for multiloop systems by cutting the loops at joints and exposing unknown constraint forces. A new development is given of the intertwining of the effects of the constraint forces on the accelerations of the bodies, with constraint force contributions requiring a two-stage update. Explicit expression of the accelerations in terms of the constraint forces leads to a particularly simple form for the evaluation of the latter. Numerical efficiency of the algorithm is shown by examples comparing to a standard, nonrecursive formulation using customary motion variables. Examples include large-angle slewing of a flexible solar sail, flexible multiantenna spacecraft with prescribed motion for internal loads calculation, a whirling chain of flexible bodies with two ends pinned, and a multiloop, flexible multibody mechanism. All the examples demonstrate the relative computational efficiency of the new formulation, with efficiency increasing with increased number of modes per flexible body.

I. Introduction

The dynamics of hinge-connected multibody systems such as machinery and spacecraft are usually simulated to extract information relevant to their design. Schiehlen [1] and Shabana [2] provide excellent reviews of the literature for systems where the bodies can be modeled as rigid or flexible, respectively. This paper concerns mechanisms made of hinge-connected elastic bodies forming structural loops, which describe numerous applications from automobiles to robotic manipulators for construction of space structures. Simulations of such mechanisms tend to require long computation times and hence reducing computer time is highly desirable in their analysis. Reduction of computer simulation time of a flexible multibody system depends broadly on formulation strategies and numerical methods used. Banerjee [3] reviewed many ways of reducing computer time by efficient formulation methods with new motion variables and using recursive or parallel algorithms for implementation. In terms of fundamental methods, Kane and Levinson [4,5] had shown that Kane's equations with a proper choice of motion variables, called generalized speeds by Kane, lead to simplest equations with the least labor. Simplicity of the final equations and reduced operation count used to generate them are directly related to computational efficiency. Lasser [6] had given a general methodology to find efficient generalized speeds for Kane's equations, whereas Mitiguy and Kane [7] gave a specific choice of efficient variables for a system of rigid bodies with rotations in revolute, Hooke's, and spherical joints. D'Eleuterio and Barfoot [8] proposed efficient variables for describing elastic motion of a single flexible body in large overall motion. For multibody systems, it has long been known [9–14] that recursive rather than direct formulations lead to faster simulations. This is attributed to the



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block-diagonal mass matrices generated by the recursive formulations, rather than the large order, dense matrix produced by direct formulations. Although computer programs for simulating flexible multibody systems with structural loops have been available for a long time, as in [15,16], recursive formulations for such systems are relatively new [12,17,18]. Bae and Haug [17] treat only rigid bodies, while Banerjee [12] and Anderson [18] consider flexible bodies. Recursive formulations for flexible body systems in the form of a tree or one with loops have all used customary motion variables of Euler angle rates for relative rotations at hinges and time derivatives of modal coordinates for elastic deformation.

The contribution of the present paper can be summarized as a recursive algorithm in efficient motion variables for flexible multibody systems with structural loops. The kernel of the formulation is the single body equations derived by Kane's method using the elastic quasi coordinates of [8]; the result is the same equations as obtained by the variational method of [8] and includes additionally a new treatment for viscous dissipation and geometric stiffness effects [19]. The next building block is a recursive formulation adapting the hinge rotation variables of [7] for a flexible multibody system in a tree configuration; the algorithm is of the same form as in [12], but with substantive changes due to the new choice of motion variables. This algorithm is then used for multiloop flexible multibody systems, by cutting the loops and exposing the unknown constraint forces. We show that the intertwining effect of the constraint forces with the accelerations requires a two-stage development: first during the backward pass and then an update in the final forward pass. It is shown that this process leads to a particularly simple acceleration-constraint relation for the evaluation of the constraint forces. We demonstrate the numerical efficiency of the overall algorithm with several simulations. This includes results for a large-angle slew of a flexible body (solar sail), and a flexible body system in tree topology where some degrees of freedom at a joint are prescribed, as when determining internal loads. Next, we test the constrained flexible body algorithm with a chain of bodies, like the example of [18] but treating the bodies as elastic rather than rigid. Finally, the full algorithm is tested on a flexible mechanism with multiple loops. Numerical results are compared for accuracy and computational speed against a standard formulation [20,21] based on Wang and Huston's extension of Kane's equation with Lagrange multipliers [22], and also the recursive formulation [12] using conventional motion variables. All the examples demonstrate the relative computational efficiency of the new formulation, with efficiency increasing with increasing number of modes per flexible body.

In the sequel we derive the equations for a single flexible body in Sec. II, followed by the hinge kinematics in Sec. III. In Sec. IV we derive the multiloop recursive algorithm for a specific example for clarity while the general algorithm, obtained by induction, is presented as a pseudocode in the Appendix. A discussion of the constraints, the solution method, and the simulation results are given in Secs. V, VI, and VII, respectively.

II. Single Body Equations in Efficient Variables

The inertial velocity $\boldsymbol{v}^P$ of P (Fig. 1), a generic particle P of a flexible body can be written in terms of the velocity $\boldsymbol{v}^O$ of a point O fixed in a flying reference frame j of inertial angular velocity $\boldsymbol{\omega}^j$ as follows, where $\boldsymbol{r}$ is the position vector of P from O in the undeformed configuration, and the deformation at P is given by a linear combination of mode shapes $\boldsymbol{\phi}_k$ weighted by modal coordinates q_k, $k = 1, \dots, n$:

$$\boldsymbol{v}^P = \boldsymbol{v}^O + \boldsymbol{\omega}^j \times \left(\boldsymbol{r} + \sum_{k=1}^{n} \boldsymbol{\phi}_k q_k\right) + \sum_{k=1}^{n} \boldsymbol{\phi}_k \dot{q}_k \tag{1}$$

Let us define generalized speeds [5], u_i, $i = 1, \dots, 6 + n$, using dot products denoted by "·"

$$u_i = \boldsymbol{\omega}^j \cdot \boldsymbol{b}_i \qquad , i = 1, \dots, 3 \tag{2}$$

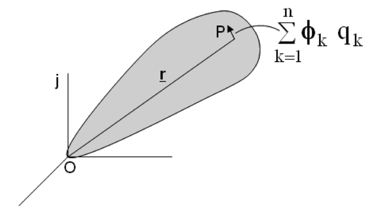

Fig. 1 Schematic of a flexible body with reference frame *j* attached to a point *O* showing undeformed position vector and deformation vector of a generic point *P*.

$$u_{3+i} = \boldsymbol{v}^O \cdot \boldsymbol{b}_i \qquad i = 1, \dots, 3 \tag{3}$$

$$\sum_{k=11}^{n} \boldsymbol{\phi}_k u_{6+k} = \boldsymbol{\omega}^j \times \sum_{k}^{n} \boldsymbol{\phi}_k q_k + \sum_{k=1}^{n} \boldsymbol{\phi}_k \dot{q}_k \tag{4}$$

Here $\boldsymbol{b}_i$ are the basis vectors of the j frame. Use of Eqs. (2–4) in Eq. (1) results in

$$\boldsymbol{v}^P = \sum_{i=1}^{3} u_{3+i} \boldsymbol{b}_i + \sum_{i=1}^{3} u_i \boldsymbol{b}_i \times \boldsymbol{r} + \sum_{k=1}^{n} \boldsymbol{\varphi}_k u_{6+k} \tag{5}$$

Comparing Eqs. (1) and (5) we note that the latter is free of modal coordinates, unlike the former. As we will see, this has the important consequence of producing constant mass matrices. By dot multiplying Eq. (4) by $\boldsymbol{\varphi}_i$ and integrating over the body we get the kinematical equations for the modal coordinates in terms of the elastic generalized speeds, $u_7, \dots, u_{6+n}$.

$$\sum_{k=1}^{n} E_{jik}(\dot{q}_k - u_{6+k}) = -\boldsymbol{\omega}^j \times \sum_{k=1}^{n} \int \boldsymbol{\phi}_k \times \boldsymbol{\phi}_i \, \mathrm{d}m q_k \tag{6}$$

where $E_{jik} = \int \boldsymbol{\phi}_i \cdot \boldsymbol{\phi}_k \, \mathrm{d}m$ are the (i, k) components of the $n \times n$ modal mass matrix E_j for body j. (We use the index j interchangeably for the body and its frame.) Equation (6), describing a coupling between rotational and modal generalized speeds, can be written in the matrix form,

$$\dot{q} = \begin{bmatrix} -E_j^{-1} G_q^j & I \end{bmatrix} \begin{Bmatrix} \omega \\ \sigma \end{Bmatrix} \tag{7}$$

where ω is a 3×1 matrix of rotational generalized speeds u_1, u_2, u_3 for body j; σ is a $n \times 1$ matrix whose elements are the elastic generalized speeds u_i, $i = 7, \dots, 6 + n$ for body j while the matrix G_q^j involving the cross product term in Eq. (6), is given by

$$G_q^j = \begin{bmatrix} \sum_{k=1}^{n} q_k \int (\tilde{\boldsymbol{\phi}}_k \boldsymbol{\phi}_i)^t \, \mathrm{d}m \\ \vdots \\ \sum_{k=1}^{n} q_k \int (\tilde{\boldsymbol{\phi}}_k \boldsymbol{\phi}_n)^t \, \mathrm{d}m \end{bmatrix} \tag{8}$$

where the "∼" denotes the usual tilde operator for matrix representation of cross products. We follow Kane's method [5] to derive the $(6 + n)$ equations of motion for the body. Generalized inertia force corresponding to the ith generalized speed u_i is given by

$$F_i^* = -\int \frac{\partial \boldsymbol{v}^P}{\partial u_i} \cdot \frac{\mathrm{d}\boldsymbol{v}^P}{\mathrm{d}t} \, \mathrm{d}m \qquad i = 1, \dots, 6 + n \tag{9}$$

where the integration is carried out over the body. Referring to Eq. (5) for evaluating the partial and total derivatives in Eq. (9), it is seen that the integrand in Eq. (9) is free of modal coordinates. In fact, the complete set of $(6 + n)$ generalized inertia forces can be written from Eq. (9) for the rigid and elastic generalized speeds, denoted with subscripts (r), and (e), respectively, in the form:

$$F_r^* = -M_1^j \begin{pmatrix} \alpha_0^j \\ a_0^j \end{pmatrix} - M_2^j \dot{\sigma}^j - X_0 \tag{10}$$

$$F_e^* = -E_j\dot{\sigma}^j - M_2^{jt}\begin{pmatrix}\alpha_0^j \\ a_0^j\end{pmatrix} - Y_0^j \tag{11}$$

Here M_1^j, M_2^j, and E^j are *constant* matrices for body j with

$$M_1^j = \begin{bmatrix} I^{j/O} & \tilde{S}^{j/O} \\ -\tilde{S}^{j/O} & m^j I \end{bmatrix} \tag{12}$$

$$M_2^j = \begin{bmatrix} \int \tilde{r}\varphi \, \mathrm{d}m \\ \int \varphi \, \mathrm{d}m \end{bmatrix} \tag{13}$$

with $I^{j/O}$, $S^{j/O}$ being, respectively, the second and first moments of inertia matrices about the origin O of coordinates of body j; I is a 3×3 unity matrix. In Eqs. (10) and (11) use has been made of the notation of [12] to represent α_0^j, a_0^j as the standard terms in the angular acceleration and acceleration, respectively, that depend on derivatives of the generalized speeds, and X_0^j, Y_0^j represent standard terms due to body forces and the inertia forces involving remainder acceleration terms free of derivatives of generalized speeds. Generalized active force due to nonconservative contact forces df at point P are given by integrating with the dot product,

$$F_i^{nc} = \int \frac{\partial \boldsymbol{v}^P}{\partial u_i} \cdot \mathrm{d}f \qquad i = 1, \ldots, 6 + n \tag{14}$$

Kane [5] has shown that once kinematical equations relating the derivatives of generalized coordinates to the generalized speeds are written as

$$\dot{q} = WU \tag{15}$$

generalized force due to forces derivable from a potential function P and a dissipation function D is given by

$$F^{cd} = -W^t \left\{ \frac{\partial P}{\partial q} + \frac{\partial D}{\partial \dot{q}} \right\} \tag{16}$$

If we take for P the sum of potential energy for mass-normalized modes of frequency ω_n and inertia-load dependent geometric stiffness (K_g) effects as in [19], and assume diagonal modal damping in forming D, that is,

$$P = 0.5q^t\omega_n^2 q + 0.5q^t\phi^t K_g \phi q \tag{17}$$

$$D = \varsigma \dot{q}^t \omega_n \dot{q} \tag{18}$$

then generalized active forces due to elasticity and damping follow from Eqs. (2), (7), and (16). For rigid body rotation and translational generalized speeds they give rise to the 6×1 matrix

$$F_r^{cd} = -\begin{bmatrix} -\left(G_q^j\right)^t \\ 0 \end{bmatrix} \left(\omega_n^2 q + 2\varsigma\omega_n \dot{q} + \phi^t K_g \phi q\right) \tag{19}$$

where the top half of the equations is simplified for the same E_j and the zero matrix on the bottom half of the coefficient matrix in Eq. (19) corresponds to translational speeds. Note that this process has yielded nonzero contributions due to elasticity and damping corresponding to rigid body rotation, just as in [8].

Generalized force due to elasticity and damping in the modal generalized speeds are given as usual by

$$F_e^{cd} = -E_j\left(\omega_n^2 q + 2\varsigma\omega_n \dot{q} + \phi^t K_g \phi q\right) \tag{20}$$

Generalized active force due only to hinge torques and forces at O and those at a node Q that may connect to an outboard body follows from Kane's method as the sum of the dot products:

$$F_i = \frac{\partial \boldsymbol{v}^O}{\partial u_i} \cdot \boldsymbol{F}^O + \frac{\partial \boldsymbol{\omega}^O}{\partial u_i} \cdot \boldsymbol{T}^O + \frac{\partial \boldsymbol{v}^Q}{\partial u_i} \cdot \boldsymbol{F}^Q + \frac{\partial \boldsymbol{\omega}^Q}{\partial u_i} \cdot \boldsymbol{T}^Q$$
$$i = 1, \ldots, 6 + n \tag{21}$$

Use of Eq. (7) for body j, and the modal matrix ψ_Q^j for elastic rotation at node Q lead to the angular velocity of a nodal rigid body at Q.

$$\omega^Q = \omega^j + \psi_Q^j \dot{q} = P_Q^j \omega^j + \psi_Q^j \sigma^j \tag{22}$$

$$P_Q^j = \left[I - \psi_Q^j E_j^{-1} G_q^j\right] \tag{23}$$

Generalized force due to hinge torques and forces at $Q(j)$ of j for rigid and elastic motion variables follow from (21)

$$F_r^h = \begin{Bmatrix} T_h^j \\ F_h^j \end{Bmatrix} + Z_{Q(j)} \begin{Bmatrix} T^{Q(j)} \\ F^{Q(j)} \end{Bmatrix}; \quad \text{where } Z_{Q(j)} = \begin{bmatrix} P_Q^{jt} & \tilde{r}^{OQ(j)} \\ 0 & I \end{bmatrix} \tag{24}$$

$$F_e^h = \Phi_{Q(j)}^t \begin{Bmatrix} T^{Q(j)} \\ F^{Q(j)} \end{Bmatrix}; \quad \text{where } \Phi_{Q(j)}^t = \begin{bmatrix} \psi^{Q(j)} & \phi^{Q(j)} \end{bmatrix} \tag{25}$$

Kane's dynamical equation [5], $F_i + F_i^* = 0$, $i = 1, .., 6 + n$, for a single, free-flying flexible body can now be written as two sets of matrix equations, after collecting terms from the above development.

$$M_1^j \begin{pmatrix}\alpha_0^j \\ a_0^j\end{pmatrix} + M_2^j \dot{\sigma}^j + X_1^j = \begin{Bmatrix} T_h^j \\ F_h^j \end{Bmatrix} + Z_{Q(j)} \begin{Bmatrix} T^{Q(j)} \\ F^{Q(j)} \end{Bmatrix} \tag{26}$$

$$E_j \dot{\sigma}^j = A_j \begin{pmatrix}\alpha_0^j \\ a_0^j\end{pmatrix} + Y_1^j + \Phi_{Q(j)}^t \begin{Bmatrix} T^{Q(j)} \\ F^{Q(j)} \end{Bmatrix} \tag{27}$$

Here the contributions from body forces, remainder acceleration terms, and the stiffness and damping terms are lumped as

$$X_1^j = X_0^j + \begin{bmatrix} -\left(G_q^j\right)^t \\ 0 \end{bmatrix} \left(\omega_n^2 q + 2\varsigma\omega_n \dot{q} + \phi^t K_g \phi q\right) \tag{28}$$

$$Y_1^j = Y_0^j - E_j\left(\omega_n^2 q + 2\varsigma\omega_n \dot{q} + \phi^t K_g \phi q\right) \tag{29}$$

III. Hinge Kinematics via Efficient Rotational Generalized Speeds

We consider two bodies, with reference frames j and $c(j)$, connected by rotational and translational joints, the body with frame $c(j)$ being inboard. Mitiguy and Kane [7] proposed generalized speeds that lead to computationally efficient equations of motion for rigid bodies undergoing rotations about revolute joints, two degrees of freedom Hooke's joints, and spherical joints. We now apply the modifications necessary for elastic bodies. For two rigid bodies j and $c(j)$ connected at Q on $c(j)$ by a revolute joint whose axis is given by h_1 in the $c(j)$ basis, Mitiguy and Kane [7] showed that an efficient rotational generalized speed u_{r1}^j is such that

$$\omega^j = C_{c(j),j}^t \left\{ \left[I - h_1 h_1^t\right] \omega^Q + u_{r1}^j h_1 \right\} \tag{30}$$

Here ω^Q is the angular velocity of a nodal rigid body at Q in $c(j)$ basis, and $C_{c(j),j}$ is a basis transformation from j to $c(j)$. Use of Eq. (22) in the above for an inboard elastic body yields,

$$\omega^j = C_{c(j),j}^t \left\{ \left[I - h_1 h_1^t\right] \left\{ P_Q^j \omega^j + \psi_Q^j \sigma^j \right\} + u_{r1}^j h_1 \right\} \tag{31}$$

When two bodies are connected by a two degrees of freedom Hooke's joint, with the first rotation about the unit vector h_1 with respect to body of frame $c(j)$ and the second rotation about the vector

h_2 expressed in the basis of frame $c(j)$, angular velocity of j includes by extension of Eq. (31) two rotational generalized speeds, u_{r1}^j, u_{r2}^j.

$$\omega^j = C_{c(j),j}^t\Big\{\Big[I - h_1 h_1^t - h_2 h_2^t\Big]\Big\{P_Q^j\omega^j + \psi_Q^j\sigma^j\Big\} + u_{r1}^j h_1 + u_{r2}^j h_2\Big\} \tag{32}$$

Generalized speeds for rotation in spherical joints are given by Eq. (2), so that

$$\boldsymbol{\omega}^j = u_{r1}\boldsymbol{j}_1 + u_{r2}\boldsymbol{j}_2 + u_{r3}\boldsymbol{j}_3 \tag{33}$$

If relative translation is allowed between point Q of body with frame $c(j)$ and $O(j)$ of body with frame j, as in a slider joint, then depending on the number of relative degree of freedom, one can introduce one, two, or three translational generalized speeds in $\delta^j = [u_{t1}, u_{t2}, u_{t3}]$,

$$u_t(i) = {}^{t(j)}\boldsymbol{v}^{O(j)}\cdot \boldsymbol{t}_i^j \qquad i = 1, 2, 3 \tag{34}$$

where $t(j)$ is a frame fixed on the flexible body node at Q on $c(j)$. The velocity of point $O(j)$ is

$$v^{O(j)} = C_{c(j),j}^t\Big\{v^{O[c(j)]} + \tilde{\omega}^{c(j)} r^{O[c(j)]Q} + \phi_Q^{c(j)}\sigma^{c(j)} + \tilde{\omega}^Q C_{c(j),t(j)}\delta^j + C_{c(j),t(j)}\dot{\delta}^j\Big\} \tag{35}$$

The expressions for angular accelerations and accelerations corresponding to the rotational and translational joints are obtained by differentiating the angular velocity and the velocity expressions in the Newtonian frame. For the purpose of developing a common notation for all types of joints we define the following relationships as in [12]. Acceleration terms involving derivatives of generalized speeds are indicated by subscript 0, and the remainder of the terms are denoted by subscript t; terms with a caret refer to inboard link nodes:

$$\begin{Bmatrix} \alpha^j \\ a^{O(j)} \end{Bmatrix} = \begin{Bmatrix} \alpha_0^j \\ a_0^{O(j)} \end{Bmatrix} + \begin{Bmatrix} \alpha_t^j \\ a_t^{O(j)} \end{Bmatrix} \tag{36}$$

$$\begin{Bmatrix} \alpha_0^j \\ a_0^{O(j)} \end{Bmatrix} = \begin{Bmatrix} \hat{\alpha}_0^j \\ \hat{a}_0^{O(j)} \end{Bmatrix} + R_j \begin{Bmatrix} \dot{u}_r^j \\ \dot{u}_t^j \end{Bmatrix} \tag{37}$$

$$\begin{Bmatrix} \hat{\alpha}_0^j \\ \hat{a}_0^{O(j)} \end{Bmatrix} = W_j \begin{Bmatrix} \alpha_0^{c(j)} \\ a_0^{O(c(j))} \end{Bmatrix} + N_j\dot{\sigma}^{c(j)} \tag{38}$$

The matrices W_j, N_j, R_j, encapsulate the transfer of kinematic information from an inboard body to its outboard body. It can be shown that the angular acceleration of frame j of a body connected by a revolute joint to an elastic body with frame $c(j)$ at a point Q of $c(j)$ and also having a translation from Q to $O(j)$ follows from Eqs. (36–38) with

$$W_j = \begin{bmatrix} C_{c(j),j}^t\Big[I - h_1 h_1^t\Big]P_Q^j & 0 \\ -C_{c(j),j}^t\Big[\tilde{r}^{O[c(j)]Q} + C_{c(j)t(j)}\tilde{\delta}^j C_{c(j)t(j)}^t P_Q^j\Big] & C_{c(j),j}^t \end{bmatrix} \tag{39}$$

$$N_j = \begin{bmatrix} C_{c(j),j}^t\Big[I - h_1 h_1^t\Big]\psi_Q^j \\ C_{c(j),j}^t\Big[\phi_Q^{c(j)} - C_{c(j)t(j)}\tilde{\delta}^j C_{c(j)t(j)}^t\psi_Q^{c(j)}\Big] \end{bmatrix} \tag{40}$$

$$R_j = \begin{bmatrix} C_{c(j),j}^t h_1 & 0 \\ 0 & C_{t(j),j}^t \end{bmatrix} \tag{41}$$

$$\alpha_t^j = C_{c(j),j}^t\Big\{\Big[I - h_1 h_1^t\Big]\Big\{P_Q^j\alpha_t^{c(j)} - D^j\Big\} + \dot{q}_{r1}^j\tilde{\omega}^L h_1\Big\} \tag{42}$$

$$\rho^j = \psi_Q^j E_{c(j)}^{-1}\dot{G}_q^j\omega^{c(j)} - \tilde{\omega}^{c(j)}\psi_Q^j\dot{q}^{c(j)} \tag{43}$$

$$\begin{aligned} a_t^{O(j)} = C_{c(j),j}^t\Big\{ & a_t^{O[c(j)]} - \tilde{r}^{O[c(j)]Q}\alpha_t^{c(j)} + \tilde{\omega}^{c(j)}\Big[\tilde{\omega}^{c(j)} r^{O[c(j)]Q} \\ & + \phi_Q^{c(j)}\sigma^{c(j)}\Big] - C_{c(j),t(j)}\tilde{\delta}^j C_{c(j),t(j)}^t\Big[P_Q^j\alpha_t^j - \rho^j\Big] \\ & + \tilde{\omega}^Q\Big[\tilde{\omega}^Q C_{c(j),t(j)}\delta^j + 2C_{c(j),t(j)}\dot{\delta}^j\Big]\Big\} \end{aligned} \tag{44}$$

The expressions for angular acceleration for a two degrees of freedom Hooke's joint and a translation joint follow similarly from Eqs. (36–38) with

$$W_j = \begin{bmatrix} C_{c(j),j}^t\Big[I - h_1 h_1^t - h_2 h_2^t\Big]P_Q^j & 0 \\ -C_{c(j),j}^t\Big[\tilde{r}^{O[c(j)]Q} + C_{c(j)t(j)}\tilde{\delta}^j C_{c(j),j}^t P_Q^j\Big] & C_{c(j),j}^t \end{bmatrix} \tag{45}$$

$$N_j = \begin{bmatrix} C_{c(j),j}^t\Big[I - h_1 h_1^t - h_2 h_2^t\Big]\psi_Q^j \\ C_{c(j),j}^t\Big[\phi_Q^{c(j)} - C_{c(j)t(j)}\tilde{\delta}^j C_{c(j),j}^t\psi_Q^{c(j)}\Big] \end{bmatrix} \tag{46}$$

$$R_j = \begin{bmatrix} C_{c(j),j}^t\big[\, h_1 \quad h_2 \,\big] & 0 \\ 0 & C_{t(j),j}^t \end{bmatrix} \tag{47}$$

$$\begin{aligned} \alpha_t^j = C_{c(j),j}^t\Big\{ & \Big[I - h_1 h_1^t - h_2 h_2^t\Big]\Big\{P_Q^j\alpha_t^{c(j)} - \rho^j\Big\} \\ & + \Big[I - h_2 h_2^t\Big]\dot{q}_{r1}^j\tilde{\omega}^Q h_1 + \dot{q}_{r2}^j\tilde{\omega}^j C_{c(j),j}^t h_2\Big\} \end{aligned} \tag{48}$$

For body j connected by a spherical joint, the angular acceleration being

$$\boldsymbol{\alpha}^j = \dot{u}_{r1}\boldsymbol{j}_1 + \dot{u}_{r2}\boldsymbol{j}_2 + \dot{u}_{r3}\boldsymbol{j}_3 \tag{49}$$

the kinematic transfer matrices of Eqs. (37) and (38) become

$$W_j = \begin{bmatrix} 0 & 0 \\ -C_{c(j),j}^t[\tilde{r}^{O[c(j)]Q} + C_{c(j)t(j)}\tilde{\delta}^j C_{c(j)t(j)}^t P_Q^j] & C_{c(j),j}^t \end{bmatrix} \tag{50}$$

$$N_j = \begin{bmatrix} 0 \\ C_{c(j),j}^t\Big[\phi_Q^{c(j)} - C_{c(j)t(j)}\tilde{\delta}^j C_{c(j)t(j)}^t\psi_Q^{c(j)}\Big] \end{bmatrix} \tag{51}$$

$$R_j = \begin{bmatrix} C_{c(j),j}^t\big[\, h_1 \quad h_2 \,\big] & 0 \\ 0 & C_{t(j),j}^t \end{bmatrix} \tag{52}$$

Obviously, in programming these equations multiplication with the zero blocks, which are particularly pervasive for spherical joints, are to be avoided.

IV. Multiloop Recursive Algorithm

The recursive algorithm given in [12] for a system of hinge-connected flexible bodies in a tree configuration can now be used after replacing the customary choice of generalized speeds and the associated kinematic transfer matrices by the efficient variables and the corresponding transfer matrices of the preceding sections. An algorithm for systems with structural loops can be derived from this algorithm by cutting a loop at a joint and imposing unknown constraint forces. The general algorithm for the case of n bodies with m loops can be obtained by induction from a specific case. Consider the system in Fig. 2a which shows five hinge-connected bodies with one closed loop and with bodies 1 and 4 hinged to the ground; Fig. 2b shows the same system after cutting the hinge connecting body 4 to the ground, and the hinge connecting bodies 2 and 5 and replacing the action of the hinges by forces and torques. Note that we have retained for generality of notation the constraint force $\boldsymbol{\lambda}_2$ which equals $-\boldsymbol{\lambda}_1$ for the structural loop.

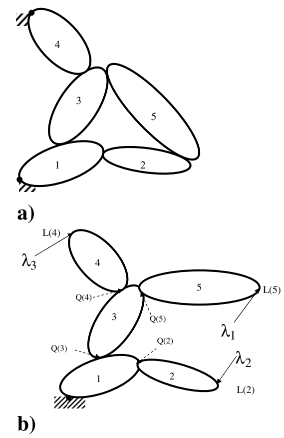

Fig. 2 a) Original constrained system. b) System derived by cutting hinges.

Backward pass: To generate the dynamic equations we first recursively form the accelerations of the hinge points and the angular accelerations of all body frames in a forward pass. Then going backward, starting with body 5, we assume a joint cut at outboard link node $L(5)$ of body 5. From Eq. (27)

$$\dot{\sigma}^5 = E_5^{-1}\left[A_5\begin{pmatrix}\alpha_0^5\\ a_0^5\end{pmatrix} + Y_1^5\right] + \hat{H}^5_{1e}\lambda_1 \tag{53}$$

$$\hat{H}^5_{1e} = E_5^{-1}\Phi^t_{L(5)} \tag{54}$$

Putting Eq. (53) in Eq. (26) yields

$$M_1^5\begin{pmatrix}\alpha_0^5\\ a_0^5\end{pmatrix} + X^5 = \begin{Bmatrix}T_h^j\\ F_h^j\end{Bmatrix} + G_{51}\lambda_1 \tag{55}$$

after making the following replacements:

$$\begin{gathered} M_1^5 \leftarrow M_1^5 + M_2^5 E_5^{-1} A_5 \qquad X^5 \leftarrow X_0^5 E_5^{-1} Y_1^5 \\ G_{51} = Z_{L(5)} - M_2^5 E_5^{-1}\Phi^t_{L(5)} \end{gathered} \tag{56}$$

Now using Eq. (37) for $j = 5$ and premultiplying the resulting equation by R_5^t yields

$$\begin{Bmatrix}\dot{u}_r^5\\ \dot{u}_t^5\end{Bmatrix} = -\nu_5^{-1}R_5^t\left[M_1^5\begin{pmatrix}\hat{\alpha}_0^5\\ \hat{a}_0^5\end{pmatrix} + X^5\right] + \nu_5^{-1}\begin{Bmatrix}\tau_h^5\\ f_h^5\end{Bmatrix} + \hat{H}^5_{1r}\lambda_1 \tag{57}$$

where

$$\nu_5 = R_5^t M_1^5 R_5 \tag{58}$$

$$\hat{H}^5_{1r} = \nu_5^{-1} R_5^t G_{51} \tag{59}$$

and τ_h^5, f_h^5 are working components of hinge torque and force. The full set of working and nonworking hinge torque and force components on body 5 are gotten by revisiting Eq. (55) and using Eq. (57) in Eq. (37),

$$\begin{Bmatrix}T_h^5\\ F_h^5\end{Bmatrix} = M_5\begin{pmatrix}\hat{\alpha}_0^5\\ \hat{a}_0^5\end{pmatrix} + X_5 - B_{31}\lambda_1 \tag{60}$$

after the following replacements have been made:

$$S_5 = I - M_1^5 R_5 \nu_5^{-1} R_5^t \tag{61}$$

$$M_5 = S_5 M_1^5 \tag{62}$$

$$X_5 = S_5 X^5 + M_1^5 R_5 \nu_5^{-1}\begin{Bmatrix}\tau_h^5\\ f_h^5\end{Bmatrix} \tag{63}$$

$$B_{31} = G_{51} - M_1^5 R_5 H^5_{1r} \tag{64}$$

Bodies 2 and 4, being terminal bodies like body 5, contribute equations exactly similar to Eqs. (53–64) with two important differences in the index for the subscripts and superscripts: body index changes from 5 to 2 or 4, and terms associated with $\boldsymbol{\lambda}_1$ such as B_{31} (first subscript for inboard body, second for constraint force) becomes B_{12} for $\boldsymbol{\lambda}_2$ and B_{33} for $\boldsymbol{\lambda}_3$; see Fig. 2b. The starting point for a body with outboard bodies, such as body 3, follows from Eq. (27), with Q(4), Q(5) being the points on body 3 at which its bodies 4 and 5 are connected:

$$E_3\dot{\sigma}^3 = A_3\begin{pmatrix}\alpha_0^3\\ a_0^3\end{pmatrix} + Y_1^3 + \Phi^t_{Q(4)}\begin{Bmatrix}T_h^4\\ F_h^4\end{Bmatrix} + \Phi^t_{Q(5)}\begin{Bmatrix}T_h^5\\ F_h^5\end{Bmatrix} \tag{65}$$

Here $\Phi_{Q(4)}$, $\Phi_{Q(5)}$ are the modal rotation matrices at $Q(4)$, $Q(5)$. Using Eqs. (60) for body 5 and its analogous equation for body 4 in Eq. (65) with subsequent use of Eq. (38) yields

$$\dot{\sigma}^3 = E_3^{-1}\left[A_3\begin{pmatrix}\alpha_0^3\\ a_0^3\end{pmatrix} + Y_1^3\right] + \hat{H}^3_{1e}\lambda_1 + \hat{H}^3_{3e}\lambda_3 \tag{66}$$

$$\hat{H}^3_{1e} = E_3^{-1}\Phi^t_{Q(5)} d_3^5 B_{31} \tag{67}$$

$$\hat{H}^3_{3e} = E_3^{-1}\Phi^t_{Q(4)} d_3^4 B_{33} \tag{68}$$

which incorporates the following replacements:

$$\begin{gathered} E_3 \leftarrow E_3 + \Phi^t_{Q(4)} d_3^4 M_4 N_4 + \Phi^t_{Q(5)} d_3^5 M_5 N_5 \\ A_3 \leftarrow A_3 - \Phi^t_{Q(4)} d_3^4 M_4 W_4 - \Phi^t_{Q(5)} d_3^5 M_5 W_5 \\ Y_1^3 \leftarrow Y_1^3 - \Phi^t_{Q(4)} d_3^4 X_4 - \Phi^t_{Q(5)} d_3^5 X_5 \end{gathered} \tag{69}$$

Equations (67–69) have used the following matrix to transfer hinge force and torque from body j to $c(j)$ for bodies 4 and 5 accounting for any translational joint.

$$d^j_{c(j)} = -\begin{bmatrix} C_{c(j),j} & C_{c(j),t(j)}\tilde{\delta}^j C_{t(j),j} \\ 0 & C_{c(j),j}\end{bmatrix} \tag{70}$$

The rotation-translation equations for body 3, modeled after Eq. (26), become

$$\begin{aligned} M_1^3\begin{pmatrix}\alpha_0^3\\ a_0^3\end{pmatrix} + M_2^3\dot{\sigma}^3 + X_1^3 &= \begin{Bmatrix}T_h^3\\ F_h^3\end{Bmatrix} - Z_{Q(4)} d_3^4\begin{Bmatrix}T_h^4\\ F_h^4\end{Bmatrix} \\ &\quad - Z_{Q(5)} d_3^5\begin{Bmatrix}T_h^5\\ F_h^5\end{Bmatrix} \end{aligned} \tag{71}$$

Substitution from Eqs. (60) and its counterpart for body 4, and the replacements

$$\begin{gathered} M_1^3 \leftarrow M_1^3 + Z_{Q(4)} d_3^4 M_4 W_4 + Z_{Q(5)} d_3^5 M_5 W_5 \\ M_2^3 \leftarrow M_2^3 + Z_{Q(4)} d_3^4 M_4 N_4 + Z_{Q(5)} d_3^5 M_5 N_5 \\ X^3 \leftarrow X^3 + Z_{Q(4)} d_3^4 X_4 + Z_{Q(5)} d_3^5 X_5 \end{gathered} \tag{72}$$

give rise to the equation

$$\begin{aligned} M_1^3\begin{pmatrix}\alpha_0^3\\ a_0^3\end{pmatrix} + M_2^3\dot{\sigma}^3 + X_1^3 &= \begin{Bmatrix}T_h^3\\ F_h^3\end{Bmatrix} + Z_{Q(4)} d_3^4 B_{33}\lambda_3 \\ &\quad + Z_{Q(5)} d_3^5 B_{11}\lambda_1 \end{aligned} \tag{73}$$

Now using Eq. (66) in Eq. (73) and making the group replacement,

$$\begin{gathered} M_1^3 \leftarrow M_1^3 + M_2^3 E_3^{-1} A_3 \qquad X^3 \leftarrow X^3 + M_2^3 E_3^{-1} Y_1^3 \\ G_{33} \leftarrow Z_3^4 - M_2^3 E_3^{-1}\Phi^t_{Q(4)} \qquad G_{31} \leftarrow Z_3^5 - M_2^3 E_3^{-1}\Phi^t_{Q(5)} \end{gathered} \tag{74}$$

produce the equations:

$$M_1^3\begin{pmatrix}\alpha_0^3\\ a_0^3\end{pmatrix} + X^3 = \begin{Bmatrix}T_h^3\\ F_h^3\end{Bmatrix} + G_{33}d_3^4B_{33}\lambda_3 + G_{31}d_3^5B_{11}\lambda_1 \tag{75}$$

Use of Eq. (37) in the above and premultiplication by R_3^t yield

$$\begin{Bmatrix}\dot u_r^3\\ \dot u_t^3\end{Bmatrix} = -\nu_3^{-1}R_3^t\left[M_1^3\begin{pmatrix}\hat\alpha_0^3\\ \hat a_0^3\end{pmatrix} + X^3\right] + \nu_3^{-1}\begin{Bmatrix}\tau_h^3\\ f_h^3\end{Bmatrix} + \hat H_{3r}^3\lambda_3 + \hat H_{1r}^3\lambda_1 \tag{76}$$

$$\hat H_{3r}^3 = \nu_3^{-1}R_3^tG_{33}d_3^4B_{33} \tag{77}$$

$$\hat H_{1r}^3 = \nu_3^{-1}R_3^tG_{31}d_3^5B_{31} \tag{78}$$

With derivatives of generalized speeds for body 3 expressed as in Eqs. (66) and (76), the inboard hinge torque and force on body 3 are obtained from Eq. (71) as

$$\begin{Bmatrix}T_h^3\\ F_h^3\end{Bmatrix} = M_3\begin{pmatrix}\hat\alpha_0^3\\ \hat a_0^3\end{pmatrix} + X_3 - B_{13}\lambda_3 - B_{11}\lambda_1 \tag{79}$$

where the following sequence of substitutions has been incorporated:

$$\begin{gathered}S_3 = I - M_1^3R_3\nu_3^{-1}R_3^t \qquad M_3 = S_3M_1^3\\ X_3 = S_3X^3 + M_1^3R_3\nu_3^{-1}\begin{Bmatrix}\tau_h^3\\ f_h^3\end{Bmatrix} \qquad B_{13} = S_3d_3^4B_{33}\\ B_{11} = S_3d_3^5B_{31}\end{gathered} \tag{80}$$

The dynamics of body 1 in Fig. 2b is influenced by hinge actions from both body 3 and body 2. With body 2 being a terminal body, derivatives of its generalized speeds are expressed just as for body 5, and its hinge loads can be written as in the development of Eq. (60) as

$$\begin{Bmatrix}T_h^2\\ F_h^2\end{Bmatrix} = M_2\begin{pmatrix}\hat\alpha_0^2\\ \hat a_0^2\end{pmatrix} + X_2 - B_{12}\lambda_2 \tag{81}$$

With body 1 being an interior body the development of its equations is again identical to that of body 3. Thus we have the sequence of substitutions:

$$\begin{gathered}E_1 \leftarrow E_1 + \Phi_{Q(2)}^td_1^2M_2N_2 + \Phi_{Q(3)}^td_1^3M_3N_3\\ A_1 \leftarrow A_1 - \Phi_{Q(2)}^td_1^2M_2W_2 - \Phi_{Q(3)}^td_1^3M_3W_3\\ Y_1^1 \leftarrow Y_1^1 - \Phi_{Q(2)}^td_1^2X_2 - \Phi_{Q(3)}^td_1^3X_3\end{gathered} \tag{82}$$

where $Q(2)$ and $Q(3)$ are points on body 1 to which bodies 2 and 3 are connected. The dynamic equations in the elastic motion variables for body 1 follow as before.

$$\dot\sigma^1 = E_1^{-1}\left[A_1\begin{pmatrix}\alpha_0^1\\ a_0^1\end{pmatrix} + Y_1^1\right] + \hat H_{1e}^1\lambda_1 + \hat H_{2e}^1\lambda_2 + \hat H_{3e}^1\lambda_3 \tag{83}$$

$$\hat H_{1e}^1 = E_1^{-1}\Phi_{Q(3)}^td_1^3B_{13} \tag{84}$$

$$\hat H_{2e}^1 = E_1^{-1}\Phi_{Q(2)}^td_1^2B_{12} \tag{85}$$

$$\hat H_{3e}^1 = E_1^{-1}\Phi_{Q(3)}^td_1^3B_{13} \tag{86}$$

Again making substitutions similar to Eq. (72),

$$\begin{gathered}M_1^1 \leftarrow M_1^1 + Z_{Q(2)}d_1^2M_2W_2 + Z_{Q(3)}d_1^3M_3W_2\\ M_2^1 \leftarrow M_2^1 + Z_{Q(2)}d_1^2M_{42}N_2 + Z_{Q(3)}d_1^3M_3N_2\\ X^1 \leftarrow X^1 + Z_{Q(2)}d_1^2X_2 + Z_{Q(3)}d_1^3X_3\end{gathered} \tag{87}$$

and those similar to Eqs. (74), namely,

$$\begin{gathered}M_1^1 \leftarrow M_1^1 + M_2^1E_1^{-1}A_1 \qquad X^1 \leftarrow X^1 + M_2^1E_1^{-1}Y_1^1\\ G_{11} \leftarrow Z_1^3 - M_2^1E_1^{-1}\Phi_{Q(3)}^t \qquad G_{12} \leftarrow Z_1^2 - M_2^1E_1^{-1}\Phi_{Q(2)}^t\\ G_{13} \leftarrow G_{11}\end{gathered} \tag{88}$$

one obtains the dynamic equations for the rigid body rotation and translation of body 1.

$$\begin{Bmatrix}\dot u_r^1\\ \dot u_t^1\end{Bmatrix} = -\nu_1^{-1}R_1^tX^1 + \nu_1^{-1}\begin{Bmatrix}\tau_h^1\\ f_h^1\end{Bmatrix} + \hat H_{1r}^1\lambda_1 + \hat H_{2r}^1\lambda_2 + \hat H_{3r}^1\lambda_2 3 \tag{89}$$

Here we have used the notations

$$\nu_1 = R_1^tM_1^1R_1 \tag{90}$$

$$\hat H_{1r}^1 = \nu_1^{-1}R_1^tG_{11}d_1^3B_{11} \tag{91}$$

$$\hat H_{2r}^1 = \nu_1^{-1}R_1^tG_{12}d_1^2B_{12} \tag{92}$$

$$\hat H_{3r}^1 = \nu_1^{-1}R_1^tG_{13}d_1^3B_{13} \tag{93}$$

Forward pass: To unscramble the dynamic equations for the rest of the bodies, we write Eq. (89) as

$$\begin{Bmatrix}\dot u_r^1\\ \dot u_t^1\end{Bmatrix} = F_{rf}^1 + H_{1r}^1\lambda_1 + H_{2r}^1\lambda_2 + H_{3r}^1\lambda_3 \tag{94}$$

where

$$F_{rf}^1 = -\nu_1^{-1}R_1^tY_2^1 + \nu_1^{-1}\begin{Bmatrix}\tau_h^1\\ f_h^1\end{Bmatrix} \tag{95}$$

$$H_{ir}^1 = \hat H_{ir}^1, \qquad i = 1, 2, 3 \tag{96}$$

If body 1 is connected to the ground, then Eq. (37) reduces to

$$\begin{Bmatrix}\alpha_0^1\\ a_0^1\end{Bmatrix} = R_1^t\begin{Bmatrix}\dot u_r^1\\ \dot u_t^1\end{Bmatrix} \tag{97}$$

Now, with the use of definitions

$$S_f^1 = R_1F_{rf}^1 \tag{98}$$

$$S_{ci}^1 = R_1H_{ir}^1, \qquad i = 1, 2, 3 \tag{99}$$

$$F_{ef}^1 = E_1^{-1}\left(Y_1^1 + A_1S_f^1\right) \tag{100}$$

Equation (83) is written as

$$\dot u_e^1 = F_{ef}^1 + H_{1e}^1\lambda_1 + H_{2e}^1\lambda_2 + H_{3e}^1\lambda_{31} \tag{101}$$

where the following update of the constraint force coefficient has been made:

$$H_{ie}^1 = \hat H_{ie}^1 + E_1^{-1}A_1S_{ci}^1, \qquad i = 1, 2, 3 \tag{102}$$

Dynamic equations for body 1, as Eqs. (94) and (101), can be used to form the equations for the rest of the bodies with the help of the kinematic transfer matrices of Sec. III. Thus, appealing to Eq. (38) for body 2 and defining

$$\gamma_f^2 = W_2S_f^1 + N_2F_{ef}^1 \tag{103}$$

$$\gamma_{ci}^2 = W_2S_{ci}^1 + N_2H_{ie}^1, \qquad i = 1, 2, 3 \tag{104}$$

one gets

$$\begin{Bmatrix} \hat{\alpha}_0^2 \\ \hat{a}_0^2 \end{Bmatrix} = \gamma_f^2 + \gamma_{c1}^2\lambda_1 + \gamma_{c2}^2\lambda_2 + \gamma_{c3}^2\lambda_3 \tag{105}$$

Initializing $\hat{H}_{ir}^2 = 0, \quad i = 1, 2, 3$ unless a nonzero value is applicable, as is the case here for $\hat{H}_{2r}^2$ we have

$$H_{ir}^2 = \hat{H}_{ir}^2 - \nu_2^{-1} R_2^t M_1^2 \gamma_{ci}^2, \qquad i = 1, 2, 3 \tag{106}$$

Derivatives of the rigid body generalized speeds for body 2 then follow with

$$F_{rf}^2 = -\nu_2^{-1} R_2^t \left(X^2 + M_2^1 \gamma_f^2 \right) + \nu_2^{-1} \begin{Bmatrix} \tau_2 \\ f_2 \end{Bmatrix} \tag{107}$$

$$\begin{Bmatrix} \dot{u}_r^2 \\ \dot{u}_t^2 \end{Bmatrix} = F_{rf}^2 + H_{1r}^2\lambda_1 + H_{2r}^2\lambda_2 + H_{3r}^2\lambda_3 \tag{108}$$

Now appealing to Eq. (37) for body 2 and forming

$$S_f^2 = \gamma_f^2 + R_2 F_{rf}^2 \tag{109}$$

$$S_{ci}^2 = \gamma_{ci}^2 + R_2 H_{ir}^2, \qquad i = 1, 2, 3 \tag{110}$$

$$\begin{Bmatrix} \alpha_0^2 \\ a_0^2 \end{Bmatrix} = S_f^2 + S_{c1}^2\lambda_1 + S_{c2}^2\lambda_2 + S_{c3}^2\lambda_3 \tag{111}$$

allow one to revisit the dynamic equations for the modal generalized speeds for body 2, initializing as before $\hat{H}_{ie}^2 = 0$, $i = 1, 2, 3$ unless a nonzero value exists, and updating

$$H_{ie}^2 = \hat{H}_{ie}^2 + E_2^{-1} A_2 S_{ci}^2, \qquad i = 1, 2, 3 \tag{112}$$

to present

$$F_{ef}^2 = E_2^{-1} \left(Y_1^2 + A_2 S_f^2 \right) \tag{113}$$

$$\dot{u}_e^{\,2} = F_{ef}^2 + H_{1e}^2\lambda_1 + H_{2e}^2\lambda_2 + H_{3e}^2\lambda_3 \tag{114}$$

Equations for rigid and elastic generalized speeds for bodies 3, 4, and 5 follow by continuing this recursion in the forward pass. The algorithmic pattern for the general case of *nb* bodies with *m* loop now emerges *by induction*. A pseudocode for this general case is given in the Appendix.

V. Motion Constraints

With dynamic equations describing the derivatives of the generalized speeds as explicitly dependent on the constraint forces, the constraint acceleration conditions can be written using Eqs. (36) and (37) in terms of these unknown derivatives and the constraint force measure numbers. The constraint equations are specific to a problem. For a loop formed by a chain of articulated flexible bodies with endpoints of body 1 and body n pinned, cutting the loop at $L(n)$ of body n and using the dynamic equations to construct the acceleration and angular acceleration of $L(n)$ yield the strikingly simple form of the acceleration constraint:

$$\left[Z_{L(n)}^t S_c^n + \Phi_{L(n)} H_e^n \right] \lambda = \begin{Bmatrix} \alpha^{L(n)} \\ a^{L(n)} \end{Bmatrix} - Z_{L(n)}^t S_f^n + \Phi_{L(n)} F_{ef}^n - \begin{Bmatrix} \alpha_t^{L(n)} \\ a_t^{L(n)} \end{Bmatrix} \tag{115}$$

The bottom three of six rows of this matrix equation corresponding to acceleration can be used to solve for the measure numbers λ of the constraint force applied on body n at its pinned end. For purposes of assigning initial conditions consistent with the constraints, one must also write the position and velocity constraints that define loop closure. The position constraints are of the functional form, $p(q) = 0$, where q are the generalized coordinates, and must be solved by Newton's method. The velocity constraint at a loop can be written as the matrix equation, $A_c(q)U = 0$, U being the column matrix of system generalized speeds. The constraint matrix $A_c(q)$ can be filled by recursively forming its entries corresponding to the generalized speeds U_j that represent the rotational, translational, and modal motion of body j, as follows, when one writes the angular velocity and velocity of the link node $L(n)$ of body n (that is constrained in rotation and/or translation).

$$\begin{Bmatrix} \omega_{L(n)} \\ v_{L(n)} \end{Bmatrix} = [Z_{L(n)} R_n \Phi_{L(n)}]\{U_n\} + Z_{L(n)}[W_n R_{c(n)} N_n]\{U_{c(n)}\} + Z_{L(n)} W_n [W_{c(n)} R_{c^2(n)} N_{c(n)}]\{U_{c^2(n)}\} + \cdots \tag{116}$$

Note that, in contrast to the recursive algorithm for constrained systems given in this paper, a nonrecursive formulation by extended Kane's equations with undetermined multipliers [22,24] is standard in the constrained dynamics literature and is expressed in the following matrix form:

$$[M(q)]\{\ddot{q}\} = \{C(q, \dot{q})\} + [A_c(q)]^t\{\lambda\}; \qquad [A_c(q)]\{\ddot{q}\} = \{D\} \tag{117}$$

Here $M(q)$ is dense, time varying, and of order equal to the degrees of freedom of the system.

VI. Numerical Solution

Numerical solutions of the equations of motion of constrained dynamic systems are prone to constraint violation and this is remedied by using differential-algebraic solvers such as DASSL described in [23]. In this work we used as an alternative, enforced constraint satisfaction [21] by overriding the solution obtained by a Kutta–Merson integrator for dependent generalized coordinates by their solution by Newton's method of the configuration constraint equations, in terms of the independent coordinates. As the motion evolves, this process is not immune to failure due to kinematic singularity, which is well documented in [24] with applicable remedies by regularization. As the emphasis in this work is a comparison of relative efficiency of formulations, singularity avoidance procedures are not used in the simulations reported later.

VII. Simulation Results

All simulation times reported here are based on a Sun Unix workstation. Large-angle slewing of a single flexible body, a solar sail spacecraft with 6 rigid body freedom and 44 elastic modes, is simulated by the above formulation and compared to the formulation [20] based on Kane's equations, and referred hereafter as the standard Kane solution, and the recursive method of [12] using customary generalized speeds (time derivatives of relative translational, rotational, and modal generalized coordinates). Results of Figs. 3 and 4 are reproduced identically by the three formulations for this system of 50 degrees of freedom for a 300 s simulation. CPU times for different formulations for a solar sail are as follows: *standard nonrecursive is 699 s; recursive for customary variables is 667 s; recursive for efficient variables is 140 s*. Figure 5 shows a system of four flexible bodies connected by revolute, Hooke's, and spherical joints. Figure 6 shows the internal torque required to prescribe zero rotation for one of the degrees of freedom of the spherical joint, by three methods, the analysis requiring a simple, prescribed motion modification of the recursive motion algorithm of [12]. CPU times for different formulations for a 10 s simulation of a 52 DOF system of four articulated flexible bodies are as follows: *standard nonrecursive is 10.2 s; recursive for customary variables is 7.6 s; and recursive for efficient variables is 5.2 s*. Comparing the computational performance of the three methods shows that the recursive formulation with the efficient generalized speeds is about twice as fast as real time in this case. The two examples described so far are obtained from the version [25] of the formulation that the present algorithm reduces to when constraints are absent. The next two examples are for constrained systems. Figure 7 gives a stroboscopic plot of a whirling chain of five flexible bodies in motion with both ends pinned. Each body has three rotational degrees of freedom and 4 or 12 elastic modes. Figure 8 shows the large-angle rotations between bodies 4 and 5, and Fig. 9 plots the first five modal coordinates of body 5; each figure overlays responses obtained with the constrained recursive formulation with efficient variables and a formulation based on [22], extended Kane's equations with Lagrange multipliers. Table 1 shows that the computational performance of the present method gets better as the number of modes per body is increased from 4 to 12. Figure 10 shows the three-dimensional flexible multiloop mechanism considered to illustrate the main algorithm of the paper. There are eight flexible bodies connected to each other by spherical joints and forming three structural loops. Comparison is made between the recursive multiloop formulation and the extended Kane formulation. The two parts in Fig. 11 show the rotations and the errors between the two solutions. The plots in Fig. 12 compare the response for eight modes and the difference between the two solutions. The top part in Fig. 13 compares the three components of the constraint force at the point where loop 3 is closed; the bottom part of Fig. 13 shows the error in loop closure for loop 3 by the shortest path and the path along the chain. It is clear that the solutions by the two methods agree very closely overall. Table 2 shows that the computational performance of the recursive algorithm with efficient variables for flexible multiloop systems gets significantly better than that of the standard, extended Kane algorithm as the number of modes per body are increased.

VIII. Conclusions

The algorithm given in this paper combines the benefits of a recursive formulation with those of an efficient choice of generalized speeds for hinge rotation and elastic motion to reduce computer simulation time for an articulated system of flexible bodies with multiloop constraints. Isolating the contribution of the constraint forces on the dynamic equations and using the intertwining effects of these forces on the accelerations of the bodies simplify the evaluation of the constraint forces. Although a systematic study of the numerical efficiency of the algorithm has not been made, results for a diverse set of articulated flexible bodies with and without structural loops show the computational efficiency of the method over standard methods. In all cases studied, computational efficiency increases with the number of modes used per flexible body. In summary, it is seen that the new algorithm reduces computer time significantly compared to the standard, nonrecursive algorithm in customary variables, for constrained systems, while giving as accurate results as the latter.

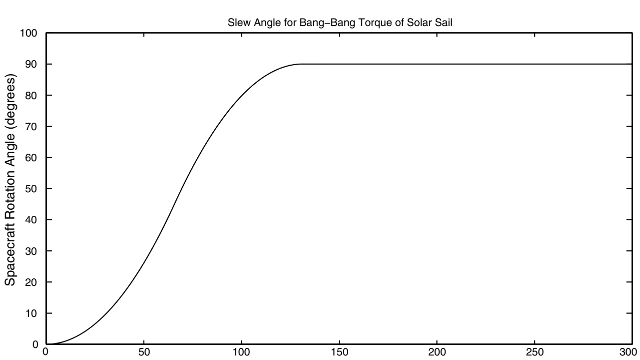

Fig. 3 Solar sail angle vs time for 90 deg slewing due to bang-bang torque.

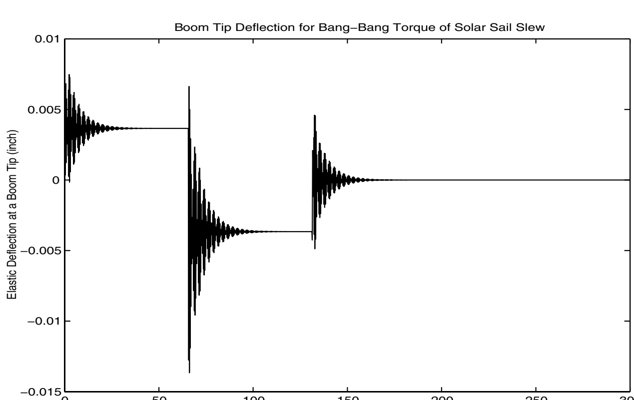

Fig. 4 Solar sail boom tip deflection with bang-bang torque.

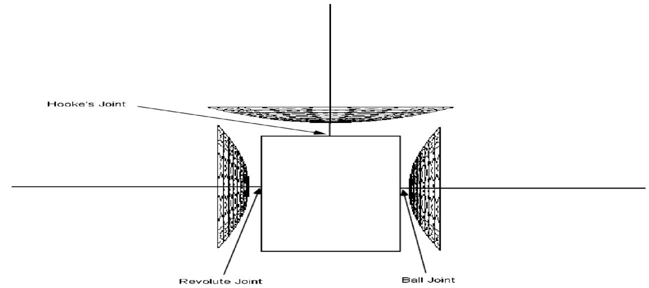

Fig. 5 Four flexible bodies with revolute, Hooke, and ball joints.

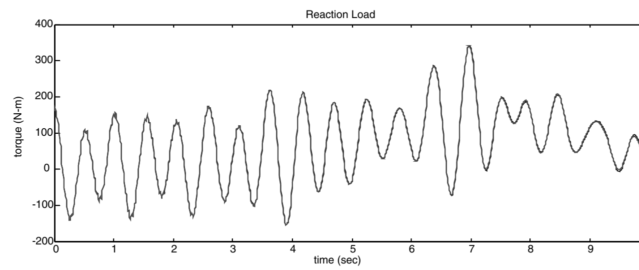

Fig. 6 Internal load corresponding to prescribed motion (locking of one rotational degree of freedom of one antenna) by three methods.

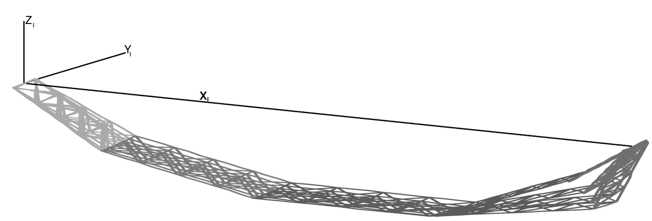

Fig. 7 Stroboscopic plot of five flexible bodies connected by spherical joints in three-dimensional, constrained whirling motion.

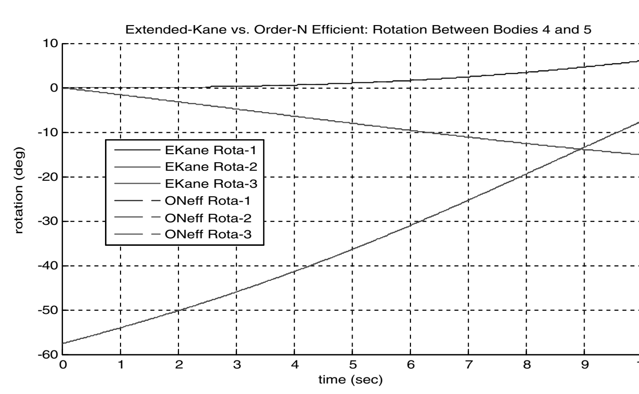

Fig. 8 Comparison of solutions for rotation in spherical joint between bodies 4 and 5 by recursive method and extended Kane's method.

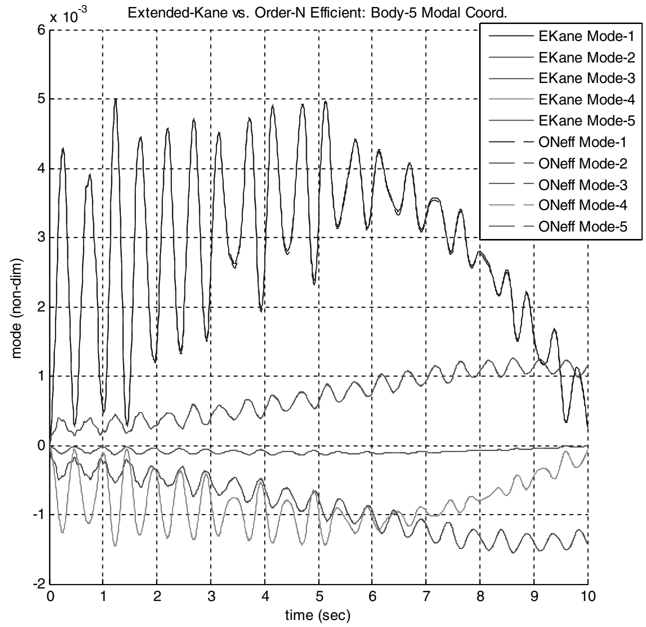

Fig. 9 Comparison of solutions for modal coordinates by recursive method and extended Kane's method.

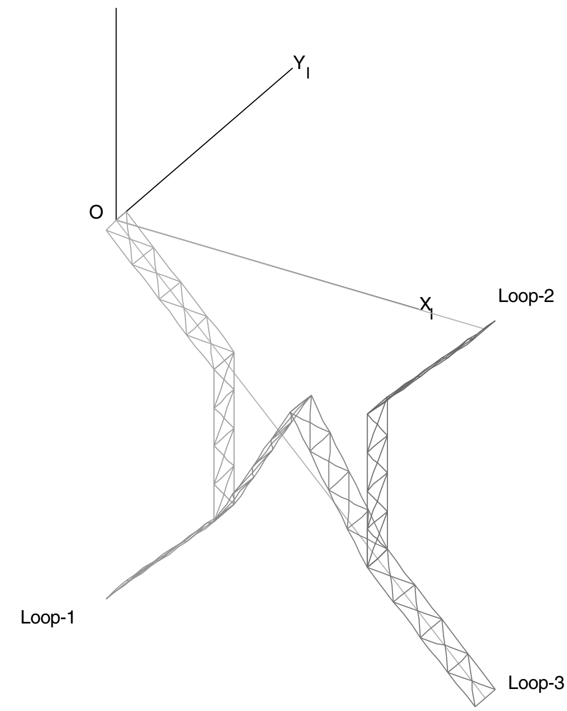

Fig. 10 Flexible mechanism with three structural loops.

Appendix: Pseudocode for the Constrained nb-Body m-loop Recursive Algorithm

Backward Pass

We give here the necessary extension beyond the standard [12] backward pass for unconstrained systems. The following segment is called during the backward pass for each body after all its mass updates are done. All bodies with cut joints become terminal bodies for which the modal mass matrix does not require an update and hence its inverse is computed only once. In the following we use the notion of a path going from a particular cut joint to body 1. Also, $Q(jp)$ is a point on a flexible body j, to which an outboard body jp is hinged.

For j going from nb to 1:

If j lies in the path for loop k

$$\hat{H}^{\,j}_{ek} = E_j^{-1}\Phi^t_{L(j)} \qquad k = 1,\ldots,m \tag{A1}$$

$$G_j = Z_{L(j)} - M_2^j \hat{H}^n_{ek} \qquad k = 1,\ldots,m \tag{A2}$$

$$\hat{H}^{\,n}_{rk} = \nu_j^{-1} R_j^t G_j \tag{A3}$$

$$B_{c(j)k} = G_j - M_1^j R_j \hat{H}^j_{rk} \qquad k = 1,\ldots,m \tag{A4}$$

else

$jp = o(j)$, outboard body of j and denoting

$$d_j^{jp} = \begin{bmatrix} C_{j,jp} & C_{j,t(jp)}\tilde{\delta}^{jp}C^t_{j,t(jp)} \\ 0 & C_{j,jp} \end{bmatrix} \tag{A5}$$

$$\hat{H}^{\,j}_{ek} = E_j^{-1}\Phi^t_{Q(jp)} d_j^{jp} B_{jk} \qquad k = 1,\ldots,m \tag{A6}$$

$$G_j = Z_{jp} - M_2^j E_j^{-1}\Phi^t_{Q(jp)} \tag{A7}$$

$$\hat{H}^{\,j}_{rk} = \nu_j^{-1} R_j^t G_j d_j^{jp} B_{jk} \qquad k = 1,\ldots,m \tag{A8}$$

$$B_{c(j)k} = G_j d_j^{jp} B_{jk} - M_1^j R_j \hat{H}^j_{rk} \qquad k = 1,\ldots,m \tag{A9}$$

Continue

Table 1 CPU comparison for constrained whirling flexible chain

Number of modes	Number of gen. speed	CPU sec for 10 s simulaton with extended Kane	Ratio of extended Kane over $O(n)$-Eff
4	35	40.75	1.01
12	75	128.61	2.49

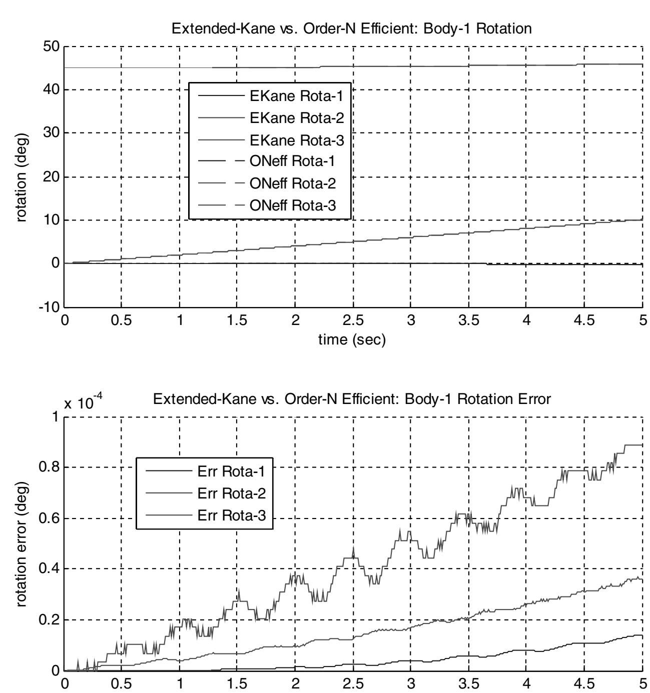

Fig. 11 Comparison of rotational angle time history for body 1 by two methods.

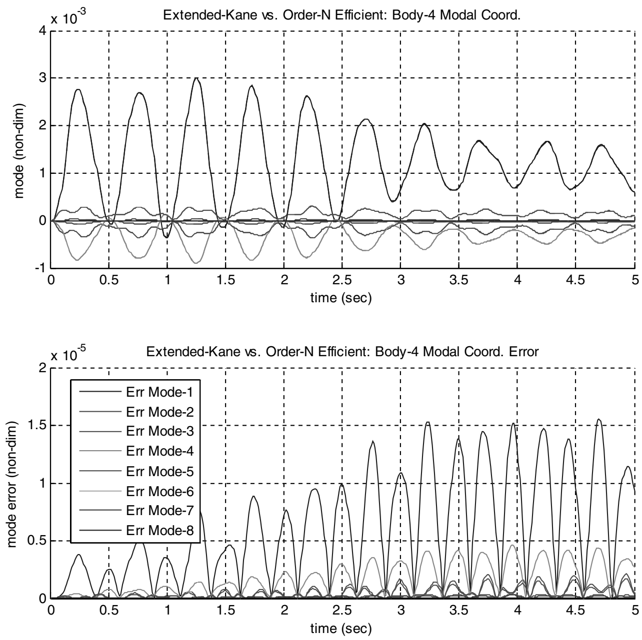

Fig. 12 Comparison of modal coordinate time history for body 4 by two methods.

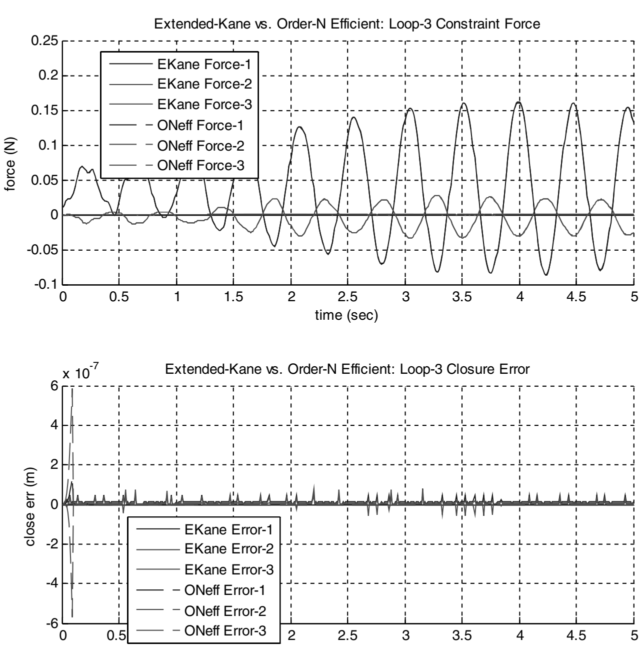

Fig. 13 Constraint force and loop closure error for loop 3.

Forward Pass

Initialize by setting $H^j_{rk} = H^j_{ek} = 0, j = 1, \ldots, nb; k = 1, \ldots, m$. Refer to Eqs. (95) and (100) for F^1_{rf}, F^1_{ef}.

$$H^1_{rk} = \hat{H}^1_{rk} \qquad k = 1, \ldots, m \tag{A10}$$

$$S^1_f = R_1 F^1_{rf} \tag{A11}$$

$$S^1_{ck} = R_1 H^1_{rk} \qquad k = 1, \ldots, m \tag{A12}$$

$$H^1_{ek} = \hat{H}^1_{ek} + E^{-1}_j A_1 S^1_{ck} \qquad k = 1, \ldots, m \tag{A13}$$

The rigid body and elastic dynamic equations for body 1 then become

Table 2 CPU time comparison for a system of eight flexible bodies connected by spherical joints with three loops and nine constraints: order-*n* efficient vs. standard extended Kane method

Number of modes	Number of gen. speeds	CPU sec for 10 s simulation using extended Kane	CPU ratio of extended Kane over order-*n* efficient method
0	24	6.28	0.969
4	56	57.57	1.516
8	88	909.29	3.053
12	120	2484.3	4.455

$$\dot{u}^1_r = F^1_{rf} + \sum_k^m H^1_{rk}\lambda_k \tag{A14}$$

$$\dot{u}^1_e = F^1_{ef} + \sum_k^m H^1_{ek}\lambda_k \tag{A15}$$

For $j = 2 \to nb$

$$\gamma^j_f = W_j S^{c(j)}_f + N_j F^{c(j)}_{ef} \tag{A16}$$

$$\gamma^j_{ck} = W_j S^{c(j)}_{ck} + N_j H^{c(j)}_{ek} \qquad k = 1, \ldots, m \tag{A17}$$

$$F^j_{rf} = -\nu^{-1}_j R^t_j \left(Y^j_1 + M^1_j \gamma^j_f\right) + \nu^{-1}_j \begin{Bmatrix} \tau_j \\ f_j \end{Bmatrix} \tag{A18}$$

$$H^j_{rk} = \hat{H}^j_{rk} - \nu^{-1}_j R^t_j M^1_j \gamma^j_{ck} \qquad k = 1, \ldots, m \tag{A19}$$

$$S^j_f = \gamma^j_f + R_j F^j_{rf} \tag{A20}$$

$$S^j_{ck} = \gamma^j_{ck} + R_j H^j_{rk} \qquad k = 1, \ldots, m \tag{A21}$$

$$F^j_{ef} = E^{-1}_j (Y^j_1 + A_j S^j_f) \tag{A22}$$

$$H^j_{ek} = \hat{H}^j_{ek} + E^{-1}_j A_j S^j_{ck} \qquad k = 1, \ldots, m \tag{A23}$$

This leads to the dynamical equations for rigid body and elastic generalized speeds for body j

$$\dot{u}^j_r = F^j_{rf} + \sum_k^m H^j_{rk}\lambda_k \tag{A24}$$

$$\dot{u}^j_e = F^j_{ef} + \sum_k^m H^j_{ek}\lambda_k \tag{A25}$$

Continue.

References

[1] Schiehlen, W., "Multibody System Dynamics: Roots and Perspectives," *Multibody System Dynamics*, Vol. 1, Kluwer Academic, Dordrecht, The Netherlands, 1997, pp. 149–188.

[2] Shabana, A., "Flexible Multibody Dynamics: Review of Past and Recent Development," *Multibody System Dynamics*, Vol. 1, Kluwer Academic, Dordrecht, The Netherlands, 1997, pp. 189–222.

[3] Banerjee, A. K., "Contributions of Multibody Dynamics to Space Flight: A Brief Review," *Journal of Guidance, Control, and Dynamics*, Vol. 26, No. 3, 2003, pp. 385–394.

[4] Kane, T. R., and Levinson, D. A., "Formulation of Equations of Motion for Complex Spacecraft," *Journal of Guidance, Control, and Dynamics*, Vol. 3, No. 2, 1980, pp. 99–112.

[5] Kane, T. R., and Levinson, D. A., *Dynamics, Theory and Application*, McGraw–Hill, New York, 1985.

[6] Lasser, M., "A Geometrical Interpretation of Kane's Equations,"

Proceedings of the Royal Society of London, Series A: Mathematical and Physical Sciences, Vol. 436, No. 1896, 1992, pp. 69–87.

[7] Mitiguy, P. C., and Kane, T. R., "Motion Variables Leading to Efficient Equations of Motion," *International Journal of Robotics Research*, Vol. 15, No. 5, 1996, pp. 522–532.

[8] D'Eleuterio, G. M. T., and Barfoot, T. D., "Just a Second, We'd Like to Go First: A First Order Discretized Formulation for Elastic Multibody Dynamics," *Proceedings of the 4th International Conference on Dynamics and Control of Structures in Space*, edited by C. L. Kirk and R. Vignjevic, Cranfield University, U.K., 24–28 May 1999.

[9] Armstrong, W. W., "Recursive Solution to the Equations of Motion of an N-Link Manipulator," *Fifth World Congress on the Theory of Machines and Mechanisms*, Vol. 2, 1979, pp. 1342–1346.

[10] Rosenthal, D. E., "An Order-n Formulation for Robotic Systems," *Journal of the Astronautical Sciences*, Vol. 38, No. 4, 1990, pp. 511–530.

[11] Jain, A., and Rodriguez, G., "Recursive Flexible Multibody System Dynamics Using Spatial Operators," *Journal of Guidance, Control, and Dynamics*, Vol. 15, No. 6, 1992, pp. 1453–1466.

[12] Banerjee, A. K., "Block-Diagonal Equations for Multibody Elastodynamics with Geometric Stiffness and Constraints," State of the Art Survey Lecture in Multibody Dynamics, European Space Agency, Noordwijk, June 1992; also *Journal of Guidance, Control, and Dynamics*, Vol. 16, No. 6, 1993, pp. 1092–1100.

[13] Pradhan, S., Modi, V. J., and Misra, A. K., "Order-N Formulation for Flexible Multibody Systems Tee Topology: Lagrangian Approach," *Journal of Guidance, Control, and Dynamics*, Vol. 20, No. 4, 1997, pp. 665–672.

[14] Featherstone, R., "A Divide-and-Conquer Articulated Body Algorithm for Parallel O(log(n)) Calculation for Rigid Body Dynamics. Part 1: Basic Algorithm," *The International Journal of Robotics Research*, Vol. 18, No. 9, 1999, pp. 867–875.

[15] Bodley, C. S., Devers, A. D., Park, A. C., and Frisch, H. P., "A Digital Computer Program for the Dynamic Interaction Simulation of Controls and Structures (DISCOS)," NASA TP 1219, Vol. 1, May 1978.

[16] ADAMS Users Manual, Ver. 2005 r2, MSC Software Corporation, 2 MacArthur Place, Santa Ana, CA 92707.

[17] Bae, D. S., and Haug, E. J., "A Recursive Formulation for Constrained Mechanical System Dynamics: Part 2," *Mechanisms, Structures, and Machines*, Vol. 15, No. 4, 1987–88, pp. 481–506.

[18] Anderson, K. S., "An Efficient Formulation for the Modeling of General Multi-Flexible-Body Constrained Systems," *International Journal of Solids and Structures*, Vol. 30, No. 7, 1993, pp. 921–945.

[19] Banerjee, A. K., and Dickens, J. M., "Dynamics of an Arbitrary Flexible Body in Large Rotation and Translation," *Journal of Guidance, Control, and Dynamics*, Vol. 13, No. 2, 1990, pp. 221–227.

[20] Banerjee, A. K., and Lemak, M. E., "Multi-Flexible-Body Dynamics Capturing Motion-Induced Stiffness," *Journal of Applied Mechanics*, Vol. 58, Sept. 1991, pp. 766–775.

[21] Lemak, M. E., "Enforcement of Configuration Constraint Satisfaction Concurrent with the Numerical Solution of Multi-Flexible-Body Equations of Motion," Engineer's Thesis, Stanford University, May 1992.

[22] Wang, J. T., and Huston, R. L., "Kane's Equations with Undetermined Multipliers-Application of Constrained Multibody Systems," *Journal of Applied Mechanics*, Vol. 54, No. 2, 1987, pp. 424–429.

[23] Brennan, K. E., Campbell, S. L., and Petzold, L., *Numerical Solution of Initial Value Problems in Differential-Algebraic Equations*, Elsevier, New York, 1989.

[24] Amirouche, F. M. L., *Computational Methods in Multibody Dynamics*, Prentice–Hall, Englewood Cliffs, NJ, 1992, pp. 434–460.

[25] Banerjee, A. K., "Efficient Generalized Speeds in a Recursive Formulation of Flexible Multibody Dynamics," in Mechanics of the 21st Century, *Proceedings of the 21st International Congress of Theoretical and Applied Mechanics*, edited by W. Gutkowski and T. Kowalewski, Springer Publications, The Netherlands, 2004.